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Realities from GeoGebra

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Introduction

Dynamic geometry is to geometry as movies are to photographs. Enhancing its use in the classroom is one of the tasks of the European project Intergeo. We are particularly interested in achieving this through GeoGebra, a dynamic geometry software that allows the simultaneous visualization of two, or even three, *Views* (graphic, algebraic and tabular) of the same objects. This possibility allows us to observe the variations taking place in every kind of illustration when the user makes any changes. This is a very useful educational feature.

We understand that reality also has several *Views*. In the physical dimension, the meaning of reality is very obvious; nevertheless, there is also another dimension concerning the ideal but objective reality of reasoning and the construction of mental structures. Furthermore, there is a subjective reality, concerning our aesthetical perception of the world. These three meanings of reality are interrelated, not just to each person, but also conceptually. The teachers and students have to adapt to their “classroom reality”.

GeoGebra is a tool that makes possible the exploration of all these kinds of reality. Thus, the goal for this contribution will be to demonstrate how to manage to understand such a reality. The underlying method consists of providing students with GeoGebra situations that will help them gradually discover the mathematical issues involved [1].

Exporting HTML websites allows the creation of interactive didactic materials that can support the dynamic visualization and the active participation of students, both at school and at home [2].

GeoGebra enables the variation of the characteristics of every construction with such simplicity that it permits the natural observation of the variable and invariable elements and, within the first group, it also shows the analysis of the way they vary. Therefore, the student can understand briefly the nature of the relations represented in the module while the teacher can ask more specific questions about them.

Mathematics = Method

A gradual approach to the study of reality from models makes its understanding easier. Furthermore, it is a good design of the activities proposed for every model that makes the difference between its acceptance and refusal. This can be seen in the following example:

The aim of the following activity is to relate the concepts of perimeter, area and symmetry as a previous step to the study of Pythagoras' theorem.

Activity 1. Find the area of the decagon that appears in the figure, knowing that each box is one square centimetre. What strategy will be the best one to calculate it? (Fig 1)

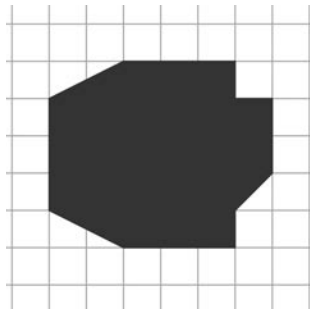


Fig 1 – A decagon

Once introduced to this problem, where the area needs to be calculated, you can suggest that students do activities designed to prove that the area of a right triangle can be deduced as long as the information of the length of its legs is provided. This can be seen in the following problem:

Activity 2. In the figure (Fig 2), there is a rectangle whose sides are 2.5 and 6 units; one of its diagonals can also be seen.

- Are all the rectangles with sides 2.5 and 6 units congruent, that is, equal in size and shape? (if one of them got cut, turned 180° and placed on top of the other, would they be exactly the same?)
- Will all their diagonals have the same length?
- Will the same happen with all rectangles with identical dimensions?

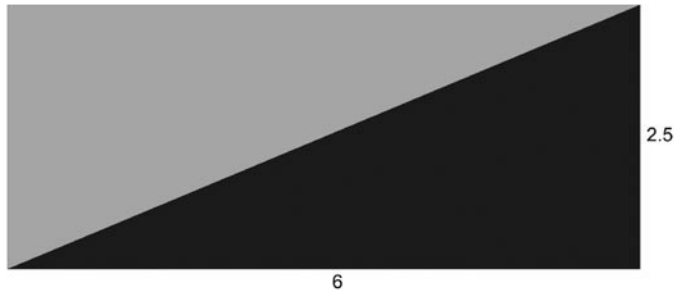


Fig 2 – Diagonal of the rectangle

Since we are already used to the beauty of a landscape full of rectangles, right triangles and areas, we are in peak conditions to suggest this activity is considered the final objective: If a right triangle is determined by the length of its legs, how can the length of the third side be deduced?

Activity 3. In the figure (Fig 3), there are four green congruent triangles and a red square.

- How long are the legs of each right-angled triangle?
- How wide is the area of the red square?
- Provided that area, can it be deduced how long is the hypotenuse of each triangle?

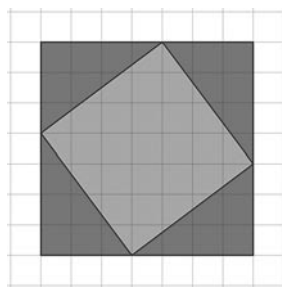


Fig 3 – Length of the hypotenuse

In the next sections, a wide range of examples will be suggested as models of the different realities. However, it is necessary to mention that, in order to make the most of some of them, the design of preparatory activities aimed at facilitating the acknowledgement of the elements in the model can be essential.

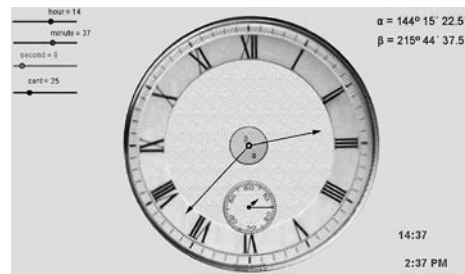
Physical reality

These are a few examples of approaches to physical reality.

Angles and proportion

An analogical clock is an excellent tool for observing different types of angles, measurement of angles, proportion, angular velocity, modular arithmetic (modules 12, 24, 60, 360°), numeral systems (decimal and sexagesimal), periodicity, arc lengths... We can set out problems regarding possible situations between the hands, coincidence, perpendicularity, the time it takes one hand to catch the other one, and so on.

The watch, which is one of the most common objects in daily



life, is a bridge between abstraction and reality (Fig 4)

Fig 4 – A normal clock



Fig 5 – A strange clock

It is a shame that the agreement for the clock direction does not match with the agreement for the positive direction of angles, although it is true that there are clocks that actually coincide, such as the one shown in the following picture, which shows one of the authors' watch (Fig 5).

Direct measurements

An image, a scale (or referent measurement) and GeoGebra: that is all the necessary in order to find out easily the measurement of lengths, distances, angles and areas on any illustration or picture. In the following aerial



view (Fig 6), the tool polygon has been used so as to have a better perspective and measure with high precision of the area of a dock.

Fig 6 – Area of a dock

Optimization

The Graphic View that GeoGebra provides works as an ideal mean to make people explore, look for solutions, and seek the hidden treasure.

In this example, the aim is to interconnect three cities through a motorway in such a way that the total length is kept to a minimum so as to save on costs. The theoretical solution is known as first Fermat Point. It can be seen in the figure (Fig 7) that the theoretical solution, where the angles formed by the rays towards the cities are the same, is not very different from the one found by engineers.

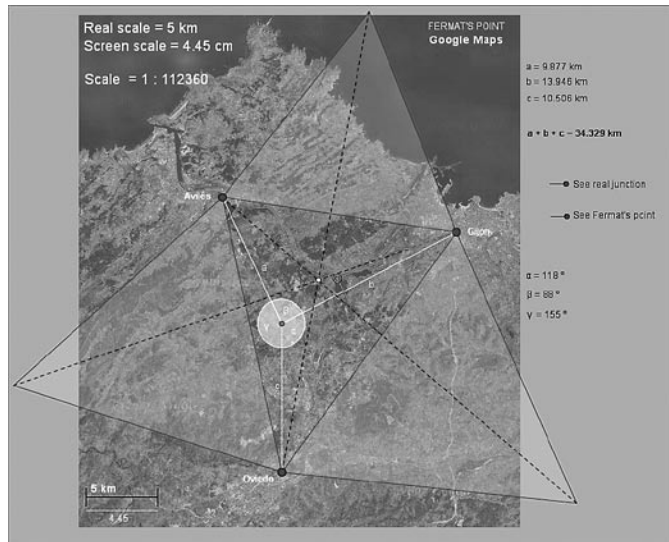
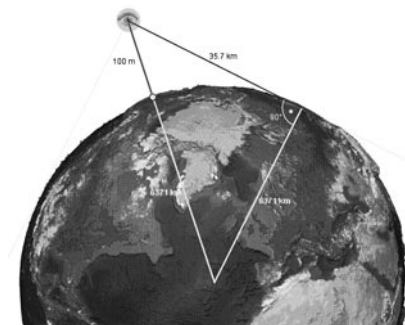


Fig 7 – Fermat Point

Schemes

A scheme can sometimes be more useful than a drawing done to scale. This happens particularly when there are big size differences between the elements that are being analysed.

The following figure (Fig 8) is a good example. The distance between the eyes of a person – placed in the figure 100 metres above the sea level – and the horizon line needs to be calculated. The elevation of the person is very small compared to the radius of the earth, therefore, in order



to create a figure that could help solve the problem, a construction should be designed in such a way that the objects are not proportionate.

Fig 8 – The horizon line

Daily observations

You do not need to go far to find similar triangles, since we contemplate our own image reduced to half by a simple mirror everyday.

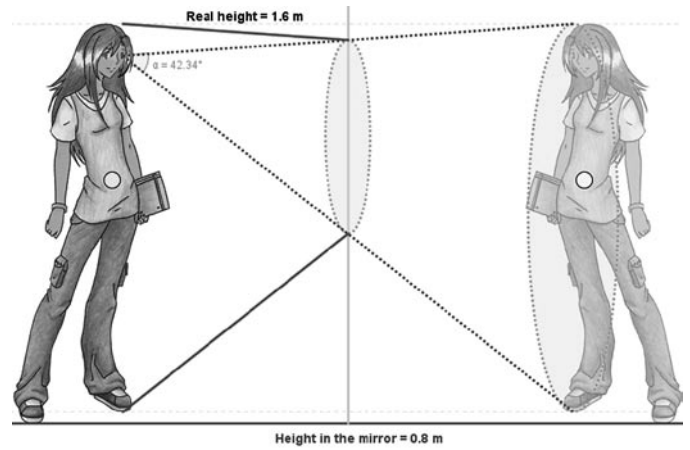


Fig 9 – Our own image

Places

The widely used GPS navigational system has a two-dimensional precedent. It is called hyperbolic navigation. Many boats still use it nowadays so as to know their position at sea. The transmitters of the signals are radio beacons placed along the coast. All emissions are synchronized, which is essential for the operation of this system. In the boat, the difference of time between the reception of the signal from one radio beacon and another can be measured and translated into a difference in the distance (Fig 10).

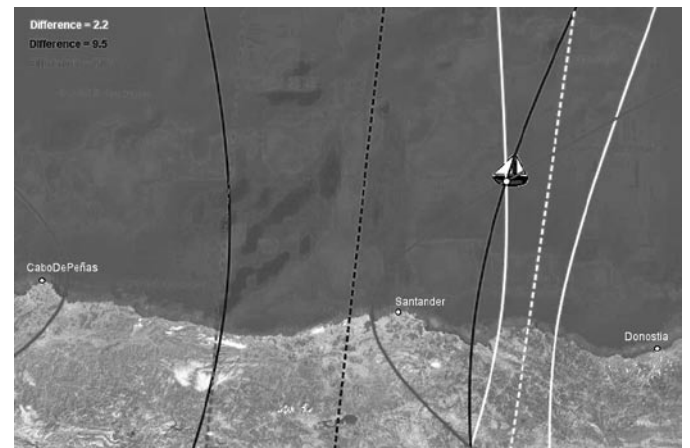


Fig 10 – Hyperbolic navigation

Logical reality

These are a few examples of approaches to logical reality.

Functional variation

When talking about variability, the most natural example could be to think about movement. The following construction is based on that analogy. The idea of movement, marked by the presence of cars, helps us to distinguish between local and global behaviour and invites us to investigate the differences in those behaviours between different kinds of functions.

The blue car describes the trajectory of the curve as if it was on a rollercoaster and, at a certain point, it derails and goes off at a tangent; this movement is represented by a green

car. All parameters can be modified within seconds: the derailment point, the coefficients and the type of function.

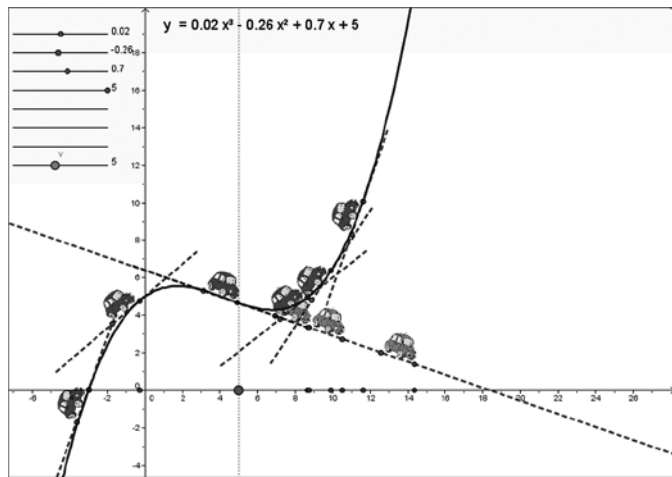


Fig 11 – Derailing

Mathematical interconnections

With GeoGebra, transformations are very dynamic. The four figures in the following image (Fig 12) correspond to the same construction and show different steps of a geometric transformation that keeps the area constant.

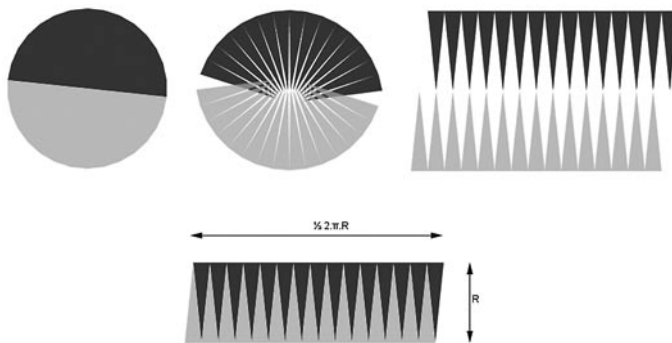
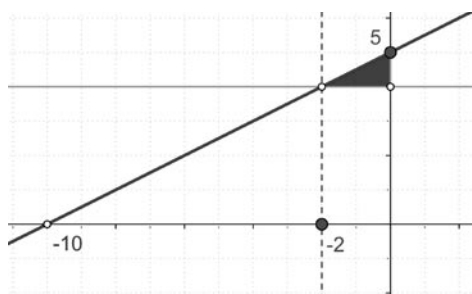


Fig 12 – Slices

In the following example, a geometrical construction that represents the product of two real numbers is presented. The relation $5(-2) = -10$ can also be represented as $-2/1 = -10/5$. It is then a proportion that can be interpreted as an



association of similarity between two triangles even respecting the rule of signs.

Fig 13 – Proportion

Projections

The construction below (Fig 14) starts with an icosahedron and allows the observation of the transformations suffered by the polyhedron when it undergoes a continuous process of truncation through flat sessions running perpendicular to the segments that link the centre of the polyhedron with the vertices. Different irregular polyhedrons start appearing, as well as a few semiregular ones, such as the truncated

icosahedron and the truncated dodecahedron. The last ones are the dual regular polyhedron of the icosahedron – the dodecahedron – and, finally, from that one, the icosahedron appears.

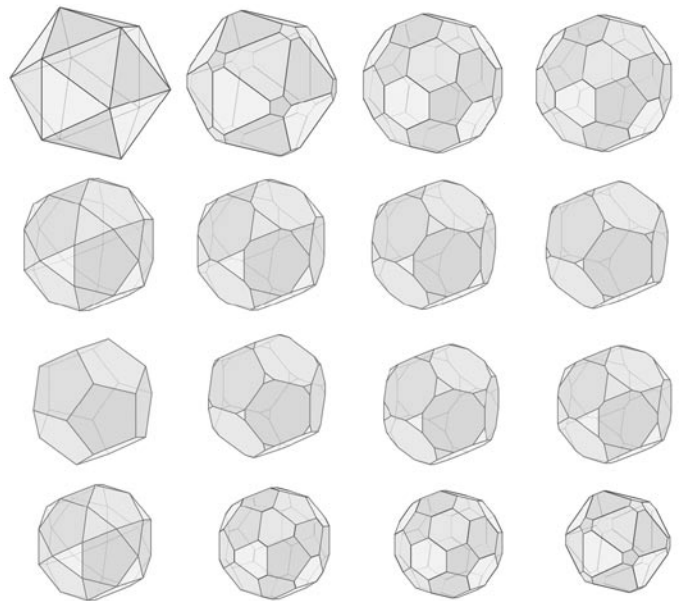
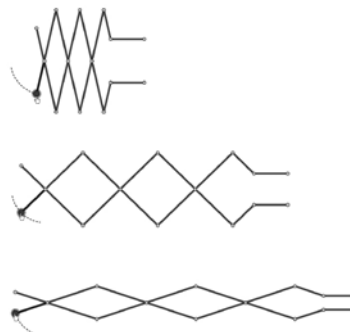


Fig 14 – Truncating

Mechanisms

In many occasions, it is possible to create models of the basic structure which allow a mechanism to achieve the aim it was designed for. These constructions are outstanding because they show the use of the geometrical properties



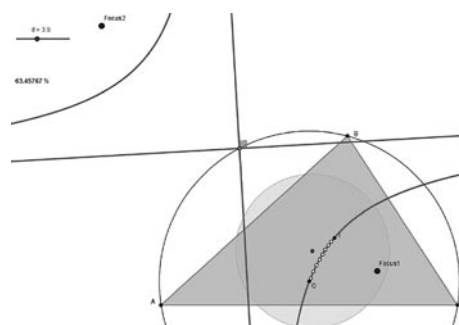
to achieve the desired operation. At the same time, the study of these properties is much more attractive for many students when they are related to a specific and known purpose (Fig 15).

Fig 15 – Mechanism

Zoooom

GeoGebra’s powerful zoom is a magnificent tool to leave a mark at exact points that form a trail of the trajectory. This procedure is quite useful so as to determine geometrical places unknown in advance.

The figure can be imagined as a trial to optimize the lighting of a triangular area with a long reach floodlight that is



perpendicular to it, such as a lantern or street light (Fig 16). In each mark, the illuminated area is the maximum.

Fig 16 – Trajectory

Perceptive reality

And finally, these are a few examples of approaches to perceptive reality.

Estimate vs measurement

In the following construction (Fig 17), both the green and yellow surfaces of the tables are congruent: they have the same shape and dimensions.

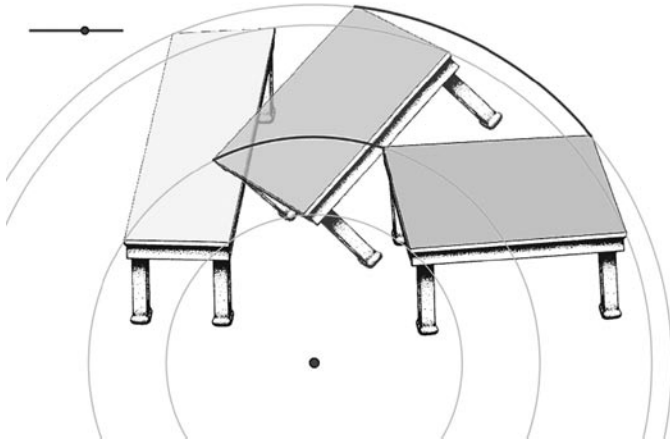


Fig 17 – Optical illusion

Construction vs drawing

Geometrical constructions are not simple drawings but graphical expressions of the variation or continuity of certain properties, depending on the constructive process used.

On the top part of the following figure (Fig 18), there are five squares which look the same, being only different in their positions and colours. However, only the one on the left hand-side is a square. That is, if the free vertices of each of them are moved, the one on the left will continue being a square, while the others are different kinds of quadrilaterals: rectangle, rhombus, rhomboid and right-angled trapezium.

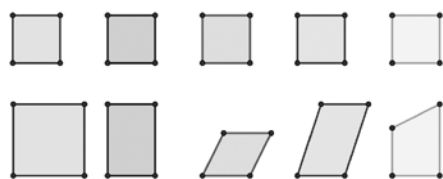


Fig 18 – A real square

Three-dimensional perception

The figure below (Fig 19) shows different points of view of the same dynamic construction. It is a replica of the logo of a Spanish art gallery.

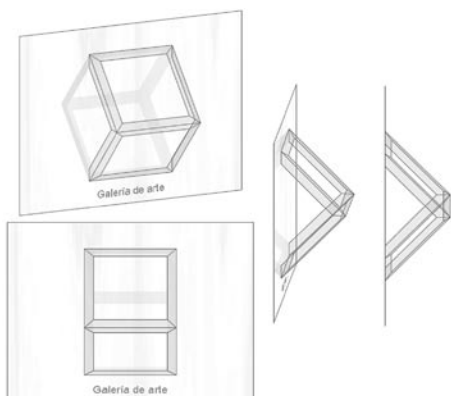


Fig 19 – Replica of a Spanish art gallery logo

Photography and cinema

In the figure below (Fig 20), the London Eye, the result of applying a homothety to a figure can be observed. The centre and the radiuses of the big wheel have also been constructed with GeoGebra.

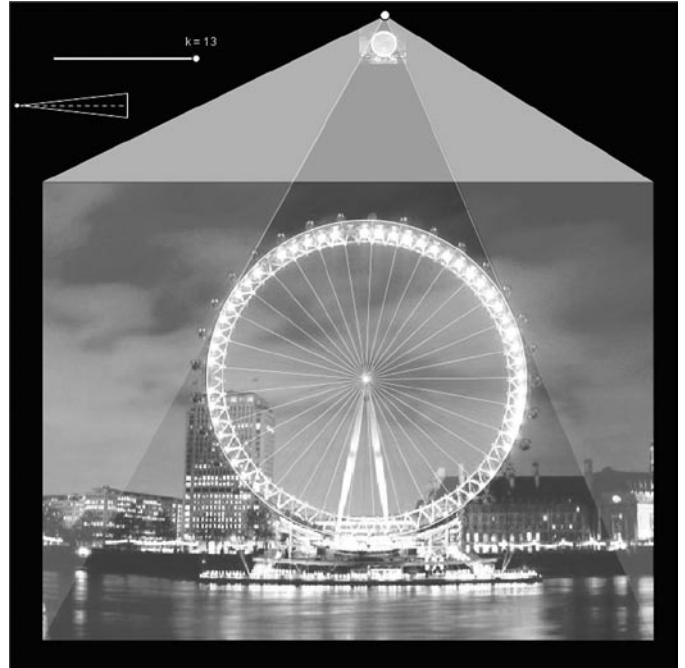


Fig 20 – London Eye

Plastic arts

In the following figure (Fig 21), the basic lines of the composition of the picture Annunciation by Leonardo da Vinci can be seen.

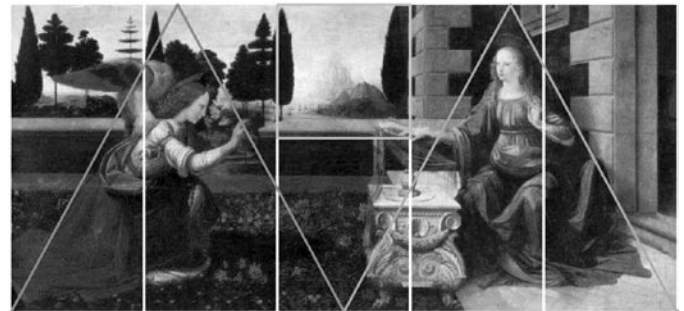


Fig 21 – Composition

Aesthetics of the movement

In the figure below (Fig 22), a snapshot of the animation produced by the vibration of a guitar string being plucked is shown.

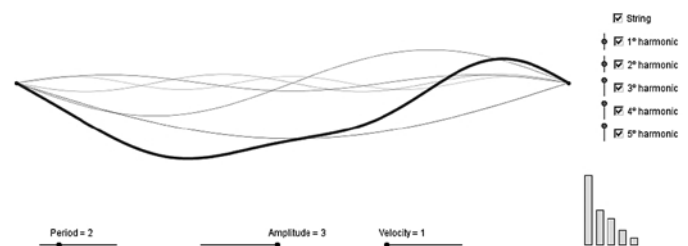


Fig 22 – Movement animation of guitar string vibration

The reality in the mathematics class

So far, the realities have been illustrated with examples through a series of mathematical situations that can be found in most of the maths curricula in secondary schools and that can be dealt with GeoGebra in class. The teacher can construct their own designs as well as take examples found on the internet, which have already been evaluated by other teachers in their classes. In other cases, it is the students themselves who build their own constructions.

As mentioned above, the dynamic geometry has been very useful with the aim to approach a mathematical concept, suggest a method to solve a problem or provide experience in different skills. Furthermore, GeoGebra may become an essential tool to use in a mathematics class and may also provoke changes in lesson organization. It can be seen in an example of investigation, starting from a very wide point of departure, that can be modelled in the class depending on the levels of the students.

We can experiment how GeoGebra has an influence in the educational practice in three dimensions: practice in class, cognitive development and the learning of attitudes.

The half



Given a square, a way of building a polygon inside it, whose area is half of it, consists of taking the middle points of two opposite sides and linking them with a segment. Investigate other procedures (Fig 23).

Fig 23 – The half

The problem, apart from being easy to tackle, contains a deliberately ambiguous question that makes one think of a series of questions: Are curves valid? Can the lines get crossed? What is understood by ‘procedure’? What is understood by ‘polygon’? Should the solution be given in a drawing, orally expressed, named by the resulting polygon or the three of them at the same time? How can it be proof that it is exactly the half of the square? What does the term ‘prove’ mean for a secondary school student? When will a proof be considered valid? And like that, one question after another can be asked, which will not be easy to answer and that both the teacher and the students must answer as if it was an actual investigation (see Fig 24 and 25).

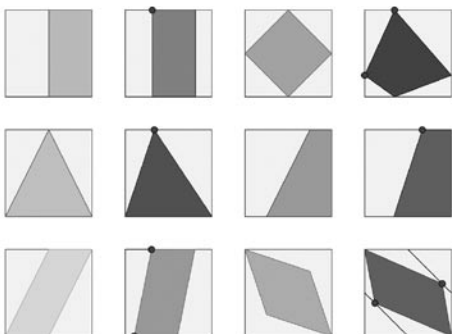


Fig 24 – Procedures

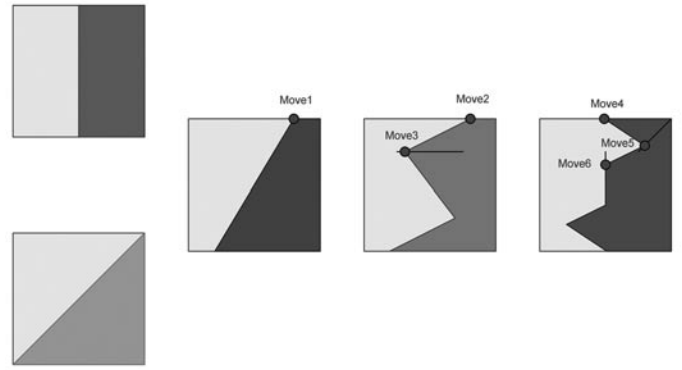


Fig 25 – Generalisations

Mosaics with half square

The search for the half of the square provides an opportunity to propose a new situation with the added point of view of movements on the plane. What would happen if we colour the obtained polygon and make copies of it as if it was a lattice (Fig 26)?

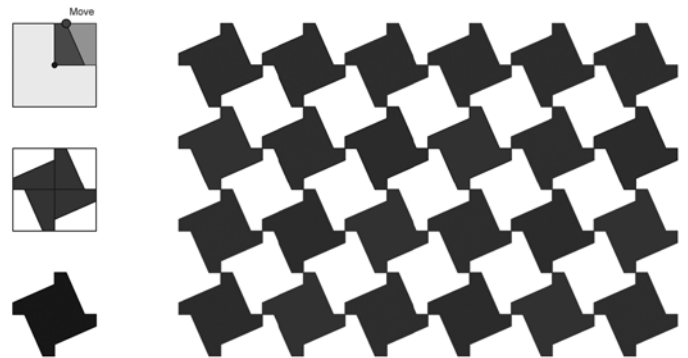


Fig 26 – Mosaic 1: copies of polygon

The use of different methods and movements gives rise to surprising solutions, such as the one found for the problem of the bone of the Alhambra Palace in Granada (Fig 27).

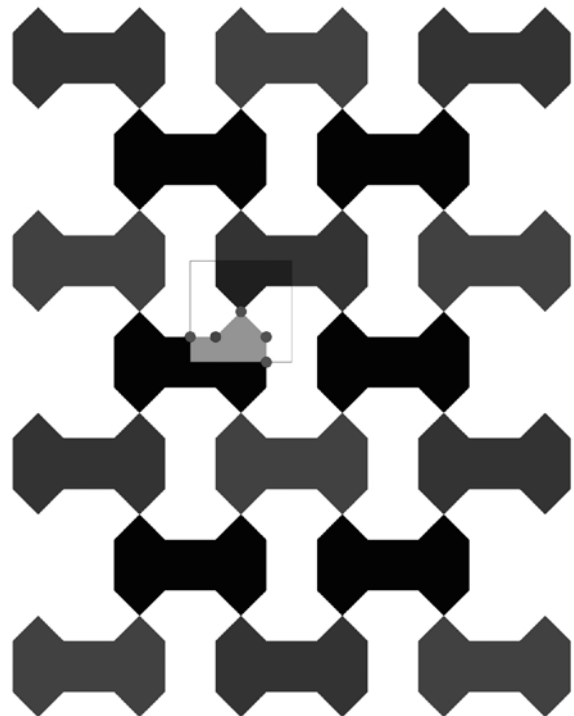


Fig 27 – Mosaic 2: Bone problem of the Alhambra Palace in Granada

By this method, we can generate other tiles that can be found in the Islamic tessellations, such as the aeroplane, a couple of stars, the needle or the leaf (Fig 28).

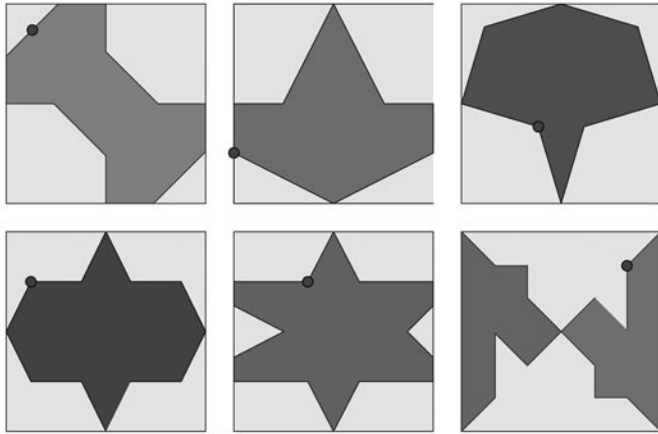


Fig 28 – Other tiles generated by GeoGebra

Mathematical art

Finally, an example that condenses in only one figure all the realities shown: physical, logical, perceptive and its development in the class. That is a very peculiar tiled floor. The possibility offered by GeoGebra to create personal tools, such as the darts and kites created by Penrose that form this design, makes the creation of constructions with repeated processes much easier.

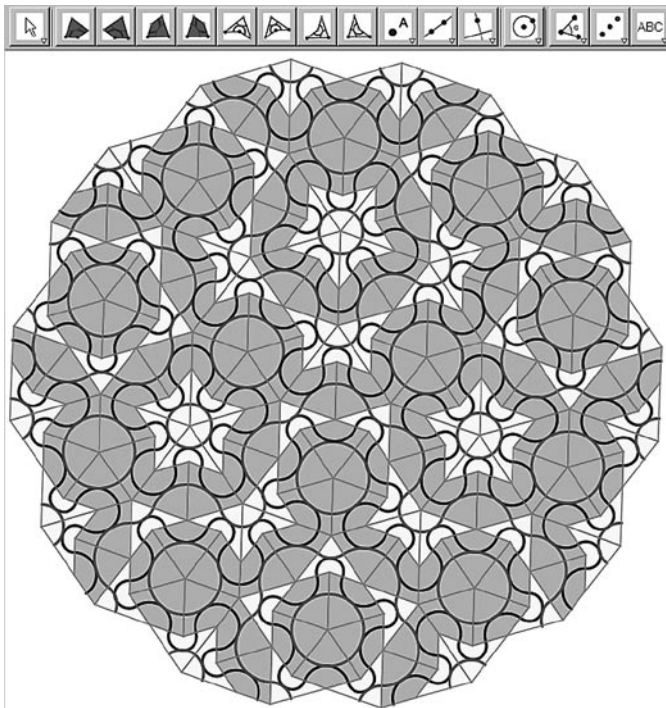


Fig 29 – Darts and kites

Conclusion

The classroom methodology will be developed both for students and teachers, who are invited to participate in a series of mathematics research situations, using GeoGebra as the main tool. For instance, as seen in "The half", the search for different methods that may allow the

construction, inside a given square, of a polygon with half its area.

Furthermore, as a collateral goal of this contribution, it emphasizes, through the different activities, several features of GeoGebra, such as the ability to include background pictures, to build up personal tools, to exploit the zoom facility, to simulate three dimensional figures using algebraic tools, to graph different algebraic functions, etc.

Also, although there are not any examples in this document, the new integrated spreadsheet and many of the new commands that GeoGebra offers will allow us to create and suggest situations related with the more and more important universes of statistics and probability.

To summarise, the use of GeoGebra enables the possibility to formulate questions that need a high level of thought. This is something that students desire and can find within themselves in order to continue enriching their experience outside school [3].

It does not actually seem crazy to consider the use of GeoGebra in class as an easy and powerful 'visual calculator'. Currently, nobody doubts that arithmetic calculations are sped up with the use of conventional calculators, which helps us concentrate on the heart of the problem instead of on the set of tedious operations that are normally involved. In the same way, geometric calculations and the study of variations between relations, basically visual and dynamic, get outstandingly sharpened with the right tools. Simply, taking a look at the *realities from GeoGebra* is enough.

Acknowledgements

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References

1. All the figures included in this paper are constructions created with GeoGebra that can be used directly on the following website: <http://geometriadinamica.es/> [Accessed 3 March 2009].
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