On the automatic discovery of Steiner-Lehmus generalizations

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(Extended Abstract)

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1 Introduction

The Theorem of Steiner-Lehmus states that if a triangle has two (internal) angle-bisectors with the same length, then the triangle must be isosceles (the converse is, obviously, also true). This is an issue which has attracted along the years a considerable interest, and we refer to [SL-URL] for a large collection of references (sometimes with comments) on this classical statement and its proof.

More recently, its generalization, regarding internal as well as external angle bisectors, has been approached through automatic tools, cf. [WL], [W04] or [B07]. Our contribution shows how, through the automatic discovery protocol of [DR], the general statements regarding internal and external bisectors are obtained. The (perhaps new) case describing the simultaneous equality of three (either internal or external) bisectors, placed on each one of the vertices, is also achieved by the same method.

The diverse generalizations of the Steiner-Lehmus theorem have been brought to our attention at a later stage of the referee process for the paper [DR]. A short summary of our results in this context, as an application of the methods described in [DR], has been appended to the original text. An extended version, but in Spanish, appeared in [LRV]. Therefore we consider of some interest presenting, to the broader audience of ADG 2010, a detailed account of our findings in this problem.

What follows, in this Extended Abstract, is a summary description, close to the text of the [DR] example.

2 Two equal bisectors

Without loss of generality we can consider a triangle of vertices $A(0,0), B(1,0)$, $C(x,y)$. Then at each vertex we can determine two bisectors (one internal, another one external) for the angles described by the lines supporting the sides of the triangle meeting at that vertex. First, we want to discover what kind of

* Supported by grant MTM2008-04699-C03-03 from the Spanish MICINN.

triangle has, say, one bisector at vertex A and one bisector at vertex B, of equal length (and, likewise, considering other pairs of vertices such as A, C and B, C). Recall that the Steiner-Lehmus Theorem states that this is the case, for internal bisectors, if and only if the triangle is isosceles. So the question here is about the equality of lengths when we consider external bisectors, too.

Algebraically we translate the construction of a bisector, say, at vertex A, as follows. We take a point (p, q) at the same distance as $C = (x, y)$ from A, so it verifies $p^2 + q^2 - (x^2 + y^2) = 0$. Then, we place this point at the line AB, by adding the equation $q = 0$. Then the midpoint from (p, q) and C will be $((x + p)/2, (y + q)/2)$ and the line defined by A and by this midpoint intersects the opposite side BC (or its prolongation) at point (a, b), verifying

$$\{p^2+q^2-(x^2+y^2) = 0, q = 0, -a(y+q)/2+b(x+p)/2 = 0, -ay+b(x-1)+y = 0\}.$$

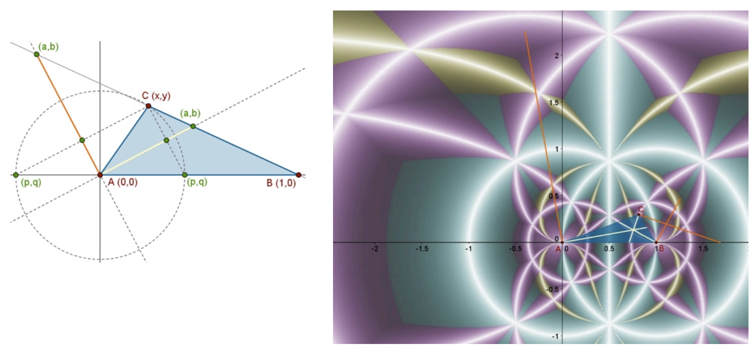
Finally, distance from (a, b) to A is given as $a^2 + b^2$, and this quantity provides the length of the bisector(s) associated to A. Notice that by placing (p, q) at different positions in the line AB, the previous construction provides both the internal and the external bisector through A. There is no way of distinguishing both bisectors, without introducing inequalities, something alien to our setting (since we work on algebraically closed fields).

Likewise, we associate a set of equations to determine the length of the bisector(s) at B, introducing a point (r, s) in the line AB, so that its distance to B is equal to that of vertex C. Then we consider the midpoint of (r, s) and C and place a line through it and B. This line intersects side AC at a point (m, n), which is defined by the following set of equations:
$\{(r - 1)^2 + s^2 - ((x - 1)^2 + y^2) = 0, s = 0, -m((y + s)/2) + n((x + r)/2 - 1) + (y + s)/2 = 0, -my + nx = 0\}$.
The length of this bisector will be $(m - 1)^2 + n^2$.

Finally, we apply the discovery protocol of [DR] to the hypotheses H given by the two sets of equations and having as thesis T the equality $(a^2 + b^2) - ((m - 1)^2 + n^2) = 0$. It is clear the that the only two (geometrically meaningful for the construction) independent variables are $\{x, y\}$, so we eliminate in $H + T$ all variables except these two, getting in this way the ideal H'. The result is a polynomial that factors as the product of y^3 (a degenerate case), $2x - 1$ (triangle is isosceles) and the degree 10 polynomial
$14x^2y^4 + y^2 + 246y^2x^6 + 76x^8 - y^6 + 8x^{10} + 9y^{10} - 164y^2x^5 + 12y^4x - 10x^2y^2 - 4x^4 - 44y^8x - 136y^4x^3 + 278y^4x^4 - 64x^7 - 164x^7y^2 + 122y^6x^2 - 6y^4 + 8x^5 - 36y^6x + 20y^2x^3 + 84y^4x^6 + 86x^4y^6 + 44x^2y^8 + 16x^6 + 41y^2x^8 + 31y^2x^4 - 40x^9 - 252y^4x^5 - 172y^6x^3 + 14y^8$
(cf.[W04], page 150, also [B07] for a picture of the curve given by this polynomial).

Next, in order to compute H'' we must choose one of the variables x, y, say, variable x, and eliminate y in the saturation of $H + H'$ by T. The result is (0), so there is no a *full set for discovering interesting conditions* (i.e. a FSDIC), according to [DR], Corollary 1. In fact it is hard to expect that for almost all triangles with vertex C placed at the locus of H' and for any interpretation of the bisectors at A and B, they will all have simultaneously an equal length. But it also means (by [DR], Proposition 2) that adding H' to the set of hypotheses, for instance, placing vertex C at any point on the degree 10 curve, there will be an interpretation for the bisectors such that the equality of lengths follow. It is easy to deduce that this is so (except for some degenerate cases) considering internal/external, external/internal and external/external bisectors (since the internal/internal case holds only for isosceles triangles). Moreover, intersecting this curve with the line $2x-1=0$ we can find out two points $x=1/2, y=(1/2)RootOf(-1+3Z^2)$ (aprox. $x=0.5000000000, y=+-0.2886751346$) where all four bisectors (the internal and external ones of A and B) have equal length. The other two points of intersection correspond to the case of equilateral triangles, where the two internal bisectors and the two infinite external bisectors of A, B have pairwise equal length, but the length is not equal for the internal and external bisectors.

A similar analysis applies when considering other pair of bisectors for vertices A, C and B, C, yielding, in each case, other, quite involved, degree 10 curves.



Construction of bisectors. Curves associated to the Steiner-Lehmus Theorem

A dynamic picture of the triangle and the resulting degree 35 (three degree 10 curves, plus two circles and one line –describing the isosceles case) curve, allowing the verification of the equality of some pairs of bisectors when vertex C is placed on some portions of the curve, has been achieved by R. Losada, using an *ad hoc* technique with the dynamic geometry software GeoGebra, cf. [LRV].

3 Three equal bisectors

A further question can be considered in this setting, regarding the equality of lengths of all three bisectors (one for each vertex) in a triangle. Here the hypotheses include the algebraic description of the bisector(s) for A, B, C and the two theses describe the equality of the lengths of the bisector(s) of A, B and of A, C. As above, we eliminate all variables except $\{x, y\}$, yielding an ideal H' generated by several polynomials (here presented as product of irreducible factors):

1. $y^3(2x - 1)(136x^2y^4 + 115021x^2y^2 - 23136x^2 + 23136x - 115021xy^2 - 136xy^4 - 21504 + 95149y^2 + 116789y^4)$,
2. $y^3(2x - 1)(17x^2 - 17x - 2 + 19y^2)(x^2 - x + 1 + y^2)$,
3. $-y^3(2x - 1)(103155x^2y^2 - 20960x^2 - 103155xy^2 + 20960x + 85459y^2 - 136y^6 - 19328 + 104787y^4)$,
4. $y^3(377084y^2x^4 - 17856 - 61088x^4 - 148192x^2 - 381436xy^4 + 87104x - 412545xy^2 + 80325y^2 + 544y^8 + 96005y^4 - 754168x^3y^2 + 122176x^3 - 2856y^6 + 789629x^2y^2 + 381436x^2y^4)$

The solution set of this system of two-variable polynomials is the x-axis (a degenerate case) plus a finite number of real and complex points. A detailed case study shows that these real points correspond to the following situations:

a) The triangle is equilateral.
b) $x = 2/17 - (2/17)RootOf(4Z^4 + 349Z^2 - 64)^2, y = RootOf(4Z^4 + 349Z^2 - 64)$ (aprox. $x = 0.09611796796, y = +- 0.4277818044$). These two points correspond to the equality of lengths for the external bisectors of A and C and the internal bisector of B.
c) Likewise, we have the two points $x = 15/17 + (2/17)RootOf(4Z^4 + 349Z^2 - 64)^2, y = RootOf(4Z^4 + 349Z^2 - 64)$ (aprox. $x = 0.09038820320, y = +- 0.4277818044$). These points correspond to the equality of lengths for the external bisectors of B and C and the internal bisector of A.
d) $x = 1/2, y = RootOf(4Z^4 - 19Z^2 - 4)$ (aprox. $x = .5000000000, y = 2.225295714$). These two points correspond to the equality of lengths for the external bisectors of A and B and the internal bisector of C.

In particular we remark that there are no triangles where two internal bisectors and one external bisector (for different vertices) have equal length, and that there are no triangles where the three external bisectors (one for each of the three vertices) are equal (except for the case of infinite length).

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