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LNCS 6785

Computational Science and Its Applications – ICCSA 2011

International Conference
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4 Part IV

Springer

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ISSN 0302-9743 e-ISSN 1611-3349
ISBN 978-3-642-21897-2 e-ISBN 978-3-642-21898-9
DOI 10.1007/978-3-642-21898-9
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011929636

CR Subject Classification (1998): C.2, H.4, F.2, H.3, D.2, C.2.4, F.1, H.5

LNCS Sublibrary: SL 1 – Theoretical Computer Science and General Issues

Typesetting: Camera-ready by author, data conversion by Scientific Publishing Services, Chennai, India

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

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Equal Bisectors at a Vertex of a Triangle

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Abstract. Given a triangle *ABC*, we study the conditions that its vertices must satisfy in order for the internal and external bisectors corresponding to one of the vertices to be equal. We investigate whether there are triangles for which the bisectors at each vertex are equal and other related properties. Automatic Deduction techniques (such as those described in [1]), implemented with CoCoA [2] and the dynamic geometry system GDI ([3], [4]), are used. Moreover, an *ad-hoc* GeoGebra [5] package has been developed (c.f. [6]) to facilitate the exploration of the problem and to improve the analysis and representation of the results in graphical form.

Keywords: Dynamic Geometry; Elementary Geometry; Automatic Deduction; Automatic Discovery; Bisectors.

1 Introduction

It is well known that if a triangle has two internal bisectors of equal length, then the triangle is isosceles (Steiner-Lehmus theorem), and this condition is also sufficient. This theorem was the subject of attention for years, as can be seen, for example, in [7]. The generalization of this result, concerning the equality of internal and external bisectors for two different vertices, was addressed recently in [8], [9] or [10], using computer algebra tools. In [11] another related contribution of the authors can be found, which establishes an open problem concerning the equality of internal or external bisectors for two or three different vertices of a triangle. In that work, automatic deduction tools in geometry, such as those described in [12], in a context of multiple theses where usual techniques are not available, were successfully applied for the first time.

In this new paper, we study the conditions that a triangle must satisfy in order for the internal and external bisectors corresponding to the *same* vertex to be of equal length. Its extension to the case where this property holds simultaneously on several vertices is also considered. We address, as well, some problems on the areas of the so called *bisector triangles*. The combination of three tools (GDI, GeoGebra and CoCoA) in the context of automatic discovery and its application to some challenging problems –if approached in the traditional way– is perhaps the main contribution of this work.

B. Murgante et al. (Eds.): ICCSA 2011, Part IV, LNCS 6785, pp. 328–341, 2011.

2 A Short Introduction to Automatic Discovery

Automatic discovery of elementary geometry theorems, although less known than automatic proving is not new. Finding the geometric locus of a point defined through some geometric constraints (say, finding the locus of a point when its projection on the three sides of a given triangle form a triangle of given constant area) can be considered as a task for the "automatic derivation" of properties approach, circa 25 years old.

Although "automatic derivation" (or locus finding) aims to discover some new geometric statements, such as "given this construction and these constraints, point P lies on the curve C", it is not exactly the same as "automatic discovery", that searches for complementary hypotheses for a (perhaps generally false) geometric statement to become true (such as stating that the three feet of the altitudes for a given triangle form an equilateral triangle and finding out what kind of triangles accomplish it). If suitably interpreted (for instance, considering a trivial thesis $0 = 0$ and searching for the conditions to verify it in terms of some specific data, such as the coordinates of point P), automatic discovery tools might as well achieve automatic derivation of properties.

The essential idea behind the different approaches to discovery is, essentially, to consider that the necessary and sufficient conditions that one needs to add to a given collection of hypotheses for a given thesis to hold is... the thesis itself. More precisely, that of adding the conjectural theses to the collection of hypotheses, and then deriving, from the new ideal of theses plus hypotheses, some new constraints in terms of the free parameters ruling the geometric situation. This derivation is achieved by elimination of the remaining (dependent) variables.

For a toy example, consider that $x - a = 0$ is the only hypothesis, that the set of points (x, a) in this hypothesis variety is determined by the value of the parameter a, and that $x = 0$ is the (generally false) thesis. Then we add the thesis to the hypothesis, getting the new ideal $(x - a, x)$, and we observe that the elimination of x in this ideal yields the constraint $a = 0$, which is indeed the extra hypothesis we have to add to the given one $x - a = 0$, in order to have a correct statement $[x - a = 0 \wedge a = 0] \Rightarrow [x = 0]$.

Indeed, things are not so trivial. Consider, for instance, $H \Rightarrow T$, where $H = (a+1)(a+2)(b+1) \subset K[a,b,c]$ and $T = (a+b+1, c) \subset K[a,b,c]$. Take as parameters $U = \{b, c\}$, a set of dim(H)-variables, independent over H. Then the elimination of the remaining variables over $H + T$ yields $H' = (c, b^3 - b)$. But $H + H' = (a+1,b,c) \cap (a+2,b,c) \cap (a+1,b-1,c) \cap (a+2,b-1,c) \cap (b+1,c)$ does not imply T, even if we add some non-degeneracy conditions expressed in terms of the free parameters U, since T vanishes over some components, such as $(a+2, b-1, c)$ (and does not vanish over some other ones, such as $(a+1, b-1, c)$).

Bearing these difficulties in mind, an elaborated discovery procedure, with several non trivial examples, is presented in [1]. One interesting example of the power of such automatic discovery protocols, related to the Steiner-Lehmus theorem and accomplishing an original and long time conjectured result, appears in [11].

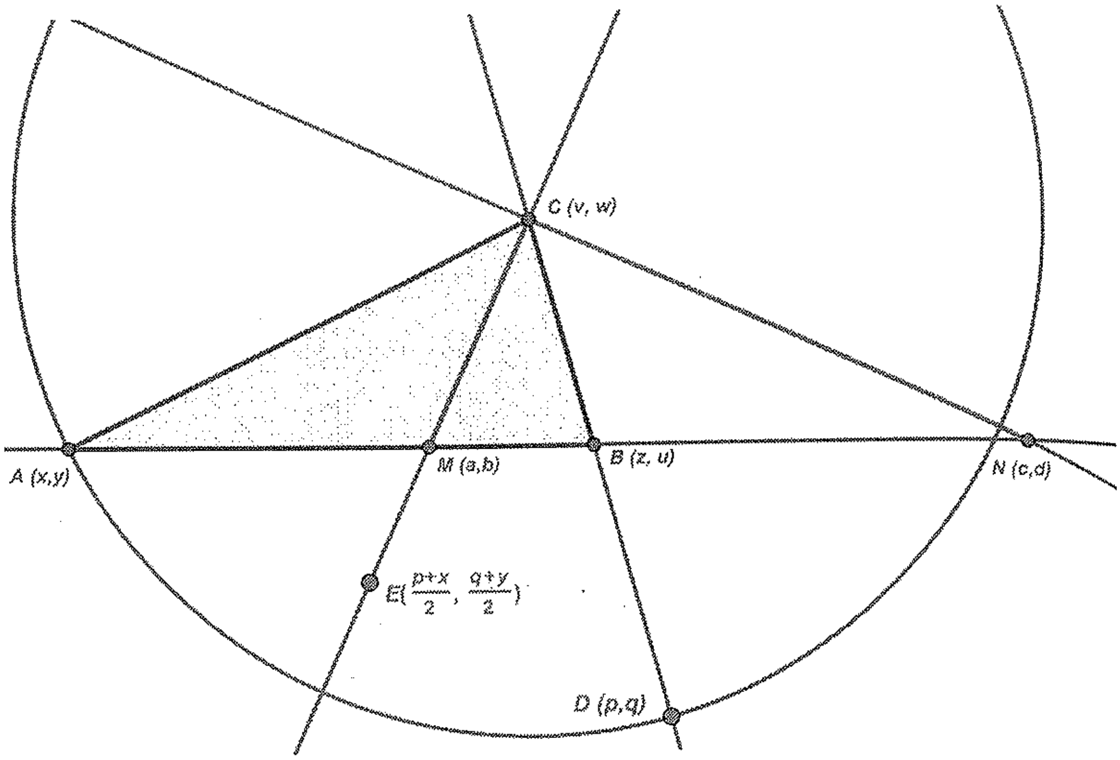

Fig. 1. Construction of the bisectors at vertex C

The discovery method of [1] has been recently revised in [12], showing that, in some precise sense, the idea of considering hypotheses and theses $H+T$ and then eliminating some variables here, is intrinsically the unique way towards discovery. In what follows we will apply, without further details, the methods described in these papers.

3 Equality of Internal and External Bisectors Corresponding to one Vertex

We start by considering the general case of a triangle with vertices $A(x,y)$, $B(z,u)$ and $C(v,w)$. First we study the conditions that A, B and C must satisfy for the two bisectors corresponding to a single vertex to be equal. We consider a vertex, say C, and determine the internal bisector and the external bisector corresponding to C. Using automatic deduction techniques, we will try to find necessary and sufficient conditions for these two bisectors to be equal. For this we translate (as done automatically by the program GDI, developed by one of the authors, see ([3], [4]) into algebra the construction of both bisectors as follows [Figure 1]:

(a) The circle with center C and radius CA which intersects side CB in $D(p,q)$ satisfies the properties:

distance(C,D)= distance(C,A) and aligned(C,B,D)

$$(v-p)^2+(w-q)^2-(v-x)^2-(w-y)^2=0 \tag{1}$$

$$Det(Mat[[v,w,1],[z,u,1],[p,q,1]]) = 0 \tag{2}$$

(b) The midpoint E of the segment DA has coordinates $\frac{p+x}{2}$ and $\frac{q+y}{2}$

(c) The bisector EC intersects the side AB at a point $M(a,b)$, which verifies the properties:

aligned(C,E,M) and aligned(A,B,M)

$$Det(Mat[[v,w,1],[\frac{p+x}{2},\frac{q+y}{2},1],[a,b,1]]) = 0 \tag{3}$$

$$Det(Mat[[x,y,1],[z,u,1],[a,b,1]]) = 0 \tag{4}$$

(d) The square of the length of the bisector CM is

$$(v-a)^2 + (w-b)^2 \tag{5}$$

(e) The algebraic conditions of the other bisector can be obtained similarly, but it is more computationally efficient the following method, based on the fact that the two bisectors are perpendicular and, therefore, the point $N(c,d)$, end of the other bisector, satisfies the following properties:

$NC \perp CE$ and aligned(A,B,N)

$$(c-v)(\frac{p+x}{2} - v) + (d-w)(\frac{q+y}{2} - w) = 0 \tag{6}$$

$$Det(Mat[[x,y,1],[z,u,1],[c,d,1]]) = 0 \tag{7}$$

(f) The square of the length of the bisector CN is:

$$(v-c)^2 + (w-d)^2 \tag{8}$$

We use now the algorithm stated in [1] (and reformulated in [12]) for the discovery of conditions under which a given property is true. We take, as set of hypotheses for the construction, H_c, the polynomials corresponding to those equations described in (1), (2), (3), (4), (6) and (7) and as thesis, T_c, take the difference (5)-(8). We add the thesis to the hypothesis set and it is clear that $H_c + T_c \Rightarrow T_c$. Since we are interested in finding conditions on the free points (vertices) in order for T_c to be satisfied, it suffices to eliminate all variables, except those of the vertices, in the ideal $H_c + T_c$. Using CoCoA we obtain a polynomial, G_C, which factors as follows:

- $F_{1C} = v^2 + w^2 - 2vx + x^2 - 2wy + y^2$, which is a degenerate circle in the variables (v,w) or (x,y), if we set the other pair of coordinates.
- $F_{2C} = (uv - ux + wx - vy - wz + yz)^2$, which is a straight line (on the side AB, AC or BC) in the variables (v,w), or (z,u) or (x,y), respectively.
- $F_{3C} = u^2v^2 + u^3w - u^2w^2 - 3u^2vx + 4uvwx + 2u^2x^2 - v^2x^2 - 3uwx^2 + w^2x^2 + vx^3 - u^3y - 2uv^2y - u^2wy + 2uw^2y + 2uvxy - 4vwxy - ux^2y + wx^2y + 2u^2y^2 + v^2y^2 - uwy^2 - w^2y^2 + vxy^2 - uy^3 + wy^3 + u^2vz - 4uvwz - u^2xz + 2v^2xz + 2uwxz - 2w^2xz - vx^2z - x^3z + 2uvyz + 4vwyz + 2wxyz - 3vy^2z - xy^2z - v^2z^2 + uwz^2 + w^2z^2 - vxz^2 + 2x^2z^2 - uyz^2 - 3wyz^2 + 2y^2z^2 + vz^3 - xz^3$, which is a

hyperbola in (v, w), or a cubic in (x, y) or (z, u), if the other coordinates are fixed.

Analogous conditions can be obtained for the vertices A and B, in terms of some corresponding polynomials G_A and G_B, so that the existence of triangles in which at least one vertex satisfies that its internal and external bisector are equal is provided by $G_A \times G_B \times G_C = 0$. A simple geometric interpretation of this result is achieved, without loss of generality, by taking two vertices in the triangle to be the origin and unit point in the X-axis. Taking $A(0,0)$ and $B(1,0)$, we get:

$G_A \times G_B \times G_C = -v^{10}w^6 - 2v^8w^8 + 2v^4w^{12} + v^2w^{14} + 4v^9w^6 + 7v^7w^8 + v^5w^{10} - 3v^3w^{12} - vw^{14} - 6v^8w^6 - 8v^6w^8 - 2w^{14} + 4v^7w^6 + 3v^5w^8 - 2v^3w^{10} - vw^{12} - v6w^6 + v^2w^{10}$ with the following factors and geometric interpretation:

- w^6, condition equivalent to the fact that the point C lies in the line AB.
- $v^2 + w^2$, representing the point A.
- $v^3 + vw^2 - v^2 + w^2$, representing a cubic (a right strophoid, see [14]) describing the locus of C for the bisectors at B to be equal.
- $v^3 + vw^2 - 2v^2 - 2w^2 + v$, representing another cubic (symmetrical to the previous one) describing the locus of C for the bisectors at A to be equal.
- $v^2 - w^2 - v$, representing a hyperbola with vertices A and B, describing the locus of C so that the bisectors of C are equal.

On [Figure 2] we have displayed the different curves representing the three geometric loci of C such that the length of the internal and external bisectors in A, B or C (respectively) coincide. This image was obtained using the dynamic color property in GeoGebra following a numerical algorithm described in [6].

This original method presents several advantages versus the straightforward approach of using the implicit plot features of some mathematical software programs, such as GDI or GeoGebra (the latter includes this feature just in the beta version 4.0, released just in August 2010, while our research on this topic started much earlier). In fact:

- implicit (or parametric) plotting is, quite often, not very reliable (as reported, for instance, in [13], see also [Figure 3]).
- implicit (or parametric) plotting requires the implicit (or parametric) equations of the given curve and this prevents, if the algebraic engine is unable to provide a suitable input, the exploration of the given geometric situation.
- plotting the geometric loci with implicit plotting does not provide additional information –as the color method does– about the behavior of parts of these loci and/or of the different regions of the plane determined by the curves, concerning the given query (in our case, the equality of bisectors).

Let us clarify these points a little, referring the reader to [6] for further details. The dynamic color method displays the curves directly from its definition as loci verifying some properties. Roughly speaking, on [Figure 2], we have considered, at each point P in the plane, the absolute values of this triple of numbers

$$(eA - iA)/(eA + iA), (eB - iB)/(eB + iB), (eC - iC)/(eC + iC)$$

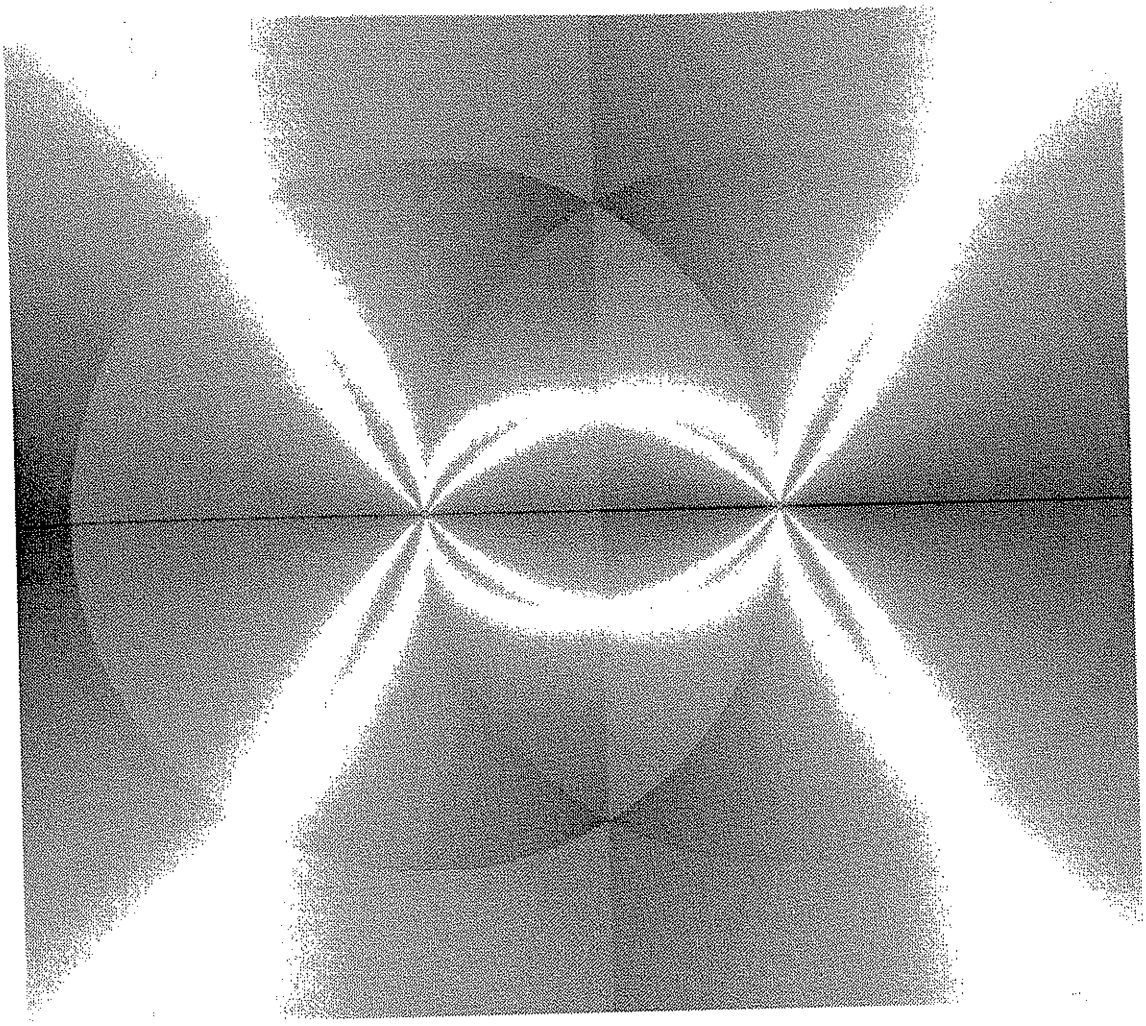

Fig. 2. Graphic result obtained for the problem of equality of internal and external bisectors at each vertex, using the procedure described in [6]. Visit http://geogebra.es/imag/fig2 for a full color figure.

Assuming that vertex C is placed at P, then eA (respectively iA) denotes the length of the external (respect. internal) bisector at A. A similar explanation applies for eB, iB, eC, iC. The three absolute values of $(eA-iA)/(eA+iA)$, $(eB-iB)/(eB+iB)$, $(eC-iC)/(eC+iC)$ are then compared and if the minimum is attained for $(eA-iA)/(eA+iA)$, then a color gradient related to red is displayed (the smaller the minimum, the lighter the color, so white lines correspond to the case of equality of internal and external bisectors). Same for $(eB-iB)/(eB+iB)$ (with a green color gradient) and for $(eC-iC)/(eC+iC)$ (with a blue color gradient). Thus, the red regions in the figure correspond to locations of C such that the relative difference of the lengths for external and internal bisectors at A is smaller than the relative difference for those at vertices B, C; and there are similar interpretations for the blue and green regions.

A similar procedure has been applied to produce [Figure 6], see the caption at this figure.

Notice that, in this automatic way, [Figure 2] displays (without having asked for it) two circles and one vertical line. It is easy to check that they correspond to

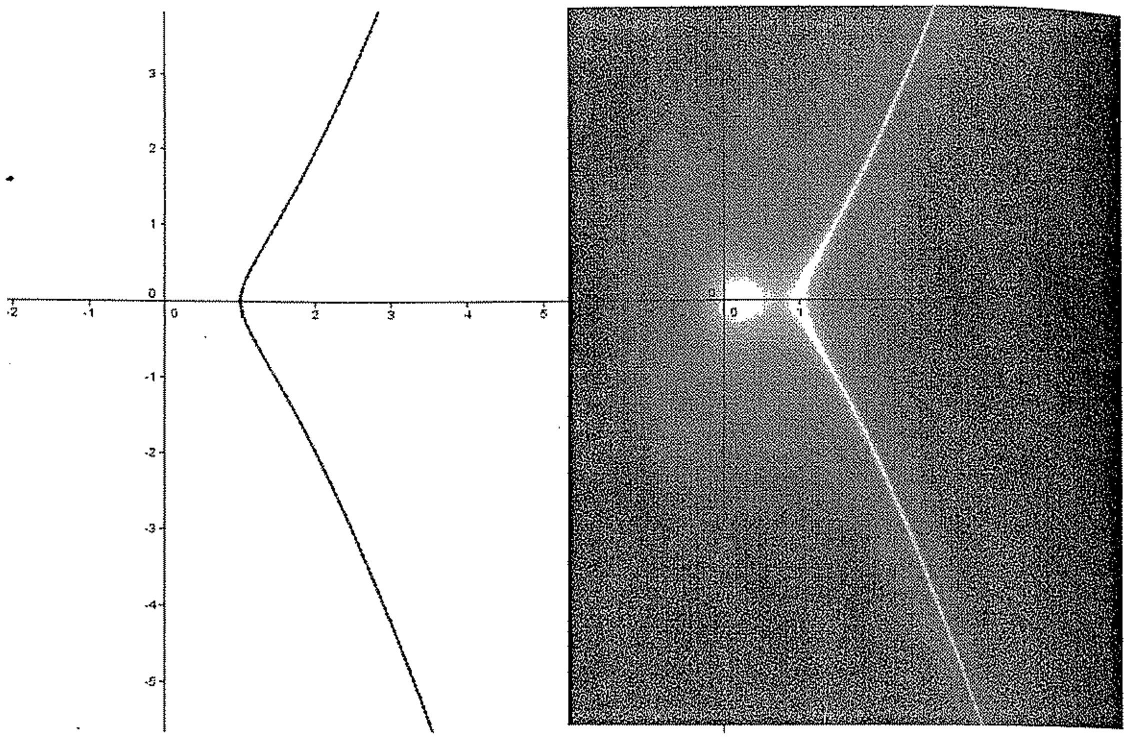

Fig. 3. Implicit plotting of the curve $x^2 + y^2 - x^3$ both with GeoGebra v.4 (left side) and with the color procedure described in [6]. Visit http://geogebra.es/imag/fig3 for a full color figure. Notice that GeoGebra misses point $(0,0)$ in the curve.

the location of C so that the triangle is isosceles. This is a case which, as we will discover later (see remarks at the end of Section 4) by algebraic means, is quite relevant in this problem. This feature is even more crucial in other contexts, such as the one described in [11], where the searched locus is an irreducible algebraic curve, but such that different branches of this single curve correspond to diverse situations concerning equality of bisectors for different vertices. In this example the dynamic color procedure turns out to be a very singular and useful tool to visually separate the different cases.

On the other hand [Figure 4] shows the solution provided by GDI for the locus set of C such that the bisectors at A have equal length. GDI first discovers the equation from the geometric conditions (using CoCoA) and then draws its graph.

We can generalize this problem by considering, instead of the equality of internal and external bisectors at a point, the case in which the length of the bisectors satisfies a given ratio k. We just have to replace the thesis T_c by $k^2((v-a)^2 + (w-b)^2) - (v-c)^2 - (w-d)^2$ and repeat the above calculations..For instance, the result obtained by specializing $A(0,0)$ and $C(1,0)$ and considering the locus of B so that we have a fixed $k-$ ratio length of bisectors at C, yields $G_C = F_{1C} \times F_{2C} \times F_{3C} = 0$, with:

- $F_{1C} = u^3k^2 + uz^2k^2 - 2u^2zk - 2z^3k - 2uzk^2 - u^3 - uz^2 - 2u^2k + 2z^2k + 2uz$
- $F_{2C} = u^3k^2 + uz^2k^2 + 2u^2zk + 2z^3k - 2uzk^2 - u^3 - uz^2 + 2u^2k - 2z^2k + 2uz$
- $F_{3C} = u^2$

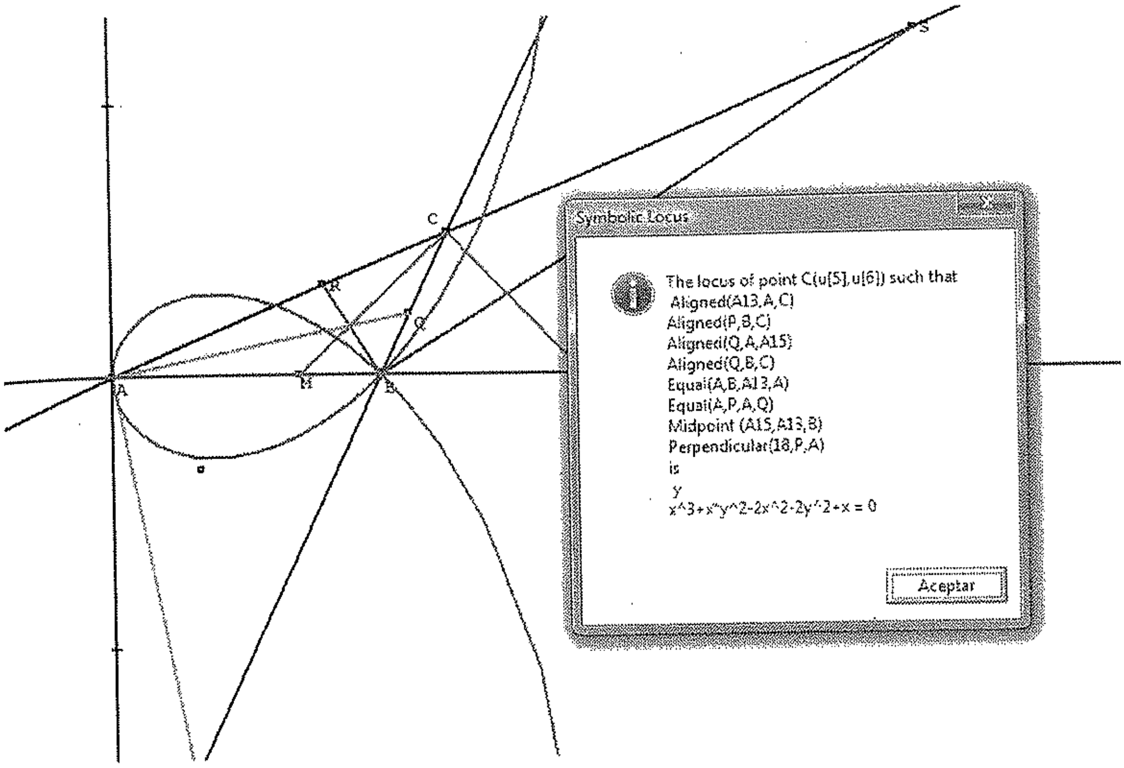

Fig. 4. Graphic and algebraic results obtained for the problem of equality of internal and external bisectors at the vertex A using GDI. Notice that the cubic curve plus a condition of degeneration ($y = 0$) are displayed. The procedure used is described in [4].

4 Equal Internal and External Bisectors at Each Vertex

The question of whether there exist triangles satisfying the equality of bisectors lengths at two and at the three vertices (i.e. such that some pair of equalities among $AQ = AP$, $BR = BS$ and $CM = CN$ hold or, even, if the three of them can simultaneously hold) arises quite naturally. Here P, Q, R, S, M, N are the feet of the corresponding bisectors.

We approach this issue by considering first the case of two vertices and, then, the full three vertices instance. Without loss of generality we assume that $A(0,0)$ and $B(1,0)$.

– Equal bisectors at C and equal bisectors at A

To the algebraic description of equal bisectors at C we add the algebraic description of the equality of the bisectors in A (introducing the corresponding variables, and so on, as in the previous Section II: (e, f) are the coordinates of intersection point between the circle with center A and radius 1 and the line AC; $Q = (g, h)$ and $P = (i, j)$), which is given by the polynomials:

$H_a = [e^2 + f^2 - 1,$
$Det(Mat([[v, w, 1], [0, 0, 1], [e, f, 1]])),$
$Det(Mat([[v, w, 1], [g, h, 1], [1, 0, 1]])),$
$Det(Mat([[0, 0, 1], [(e + 1)/2, f/2, 1], [g, h, 1]])),$

$Det(Mat([[i, j, 1], [1, 0, 1], [v, w, 1]]))$,
$i * (e + 1)/2 + j * f/2]$

$T_a = [g^2 + h^2 - i^2 - j^2]$

Eliminating all but the indeterminates v and w, we obtain:

$$w^4 - 1/2vw^2, \quad v^2w^3 - 3/2vw^3, \quad v^3w^2 - 3/2v^2w^2$$

The intersection of these three curves gives the line $w = 0$ (side AB) and the points with coordinates $\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right)$. We conclude that the only non-degenerate triangles with this property (equality of bisectors lengths at these two vertices, at least) are those with vertices $A(0,0)$, $B(1,0)$, $C\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right)$, see [Figure 5].

It could be interesting to remark that such triangles are strictly isosceles (i.e. not equilateral). In particular, this observation yields the non-existence of triangles with equal pair of bisectors in each of its three vertices. A computational proof of this fact is also easy to obtain, see next item.

Moreover, we know, by the classical Steiner-Lehmus Theorem, that isosceles triangles have two internal bisectors of equal length. And it is easy, by symmetry, to conclude that such triangles have also two external bisectors with equal length (although different, in general, from the common length of the internal bisectors).

Thus, the obtained isosceles triangles (with equal angles at C, A), namely, those with vertices $A(0,0)$, $B(1,0)$, $C\left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right)$, must have *all four* bisectors at vertices C, A with equal lengths.

That is, we conclude there are not triangles with two vertices C, A having at each of them equal lengths l_C (respectively, l_A) for the internal and external bisectors of C (respectively, of A), but with l_A different from l_C.

A similar discussion can be carried for the case of equality of bisectors at A, B.

– Equal bisectors at C, equal bisectors at A and equal bisectors at B

It suffices to calculate, for example, which triangles have equal bisectors at B and C and check if any of them is one of the found in the previous case. This calculation leads to:

$$w^5 + 3/2vw^3, \quad v^2w^3 + 1/2vw^3, \quad vw^4 + 1/2w^4, \quad v^3w^2 - v^2w^2 + 1/2w^4$$

whose associated variety is composed of the line $w = 0$ and the points with coordinates $\left(\frac{-1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

Therefore we conclude that there is no triangle with equal pair of bisectors in each of its three vertices.

5 A Related Problem

In [11] we have determined the conditions for bisectors of different vertices to have the same length. In this paper we have studied the case of equality of bisectors for a single vertex. Considering both contributions it is easy to deduce

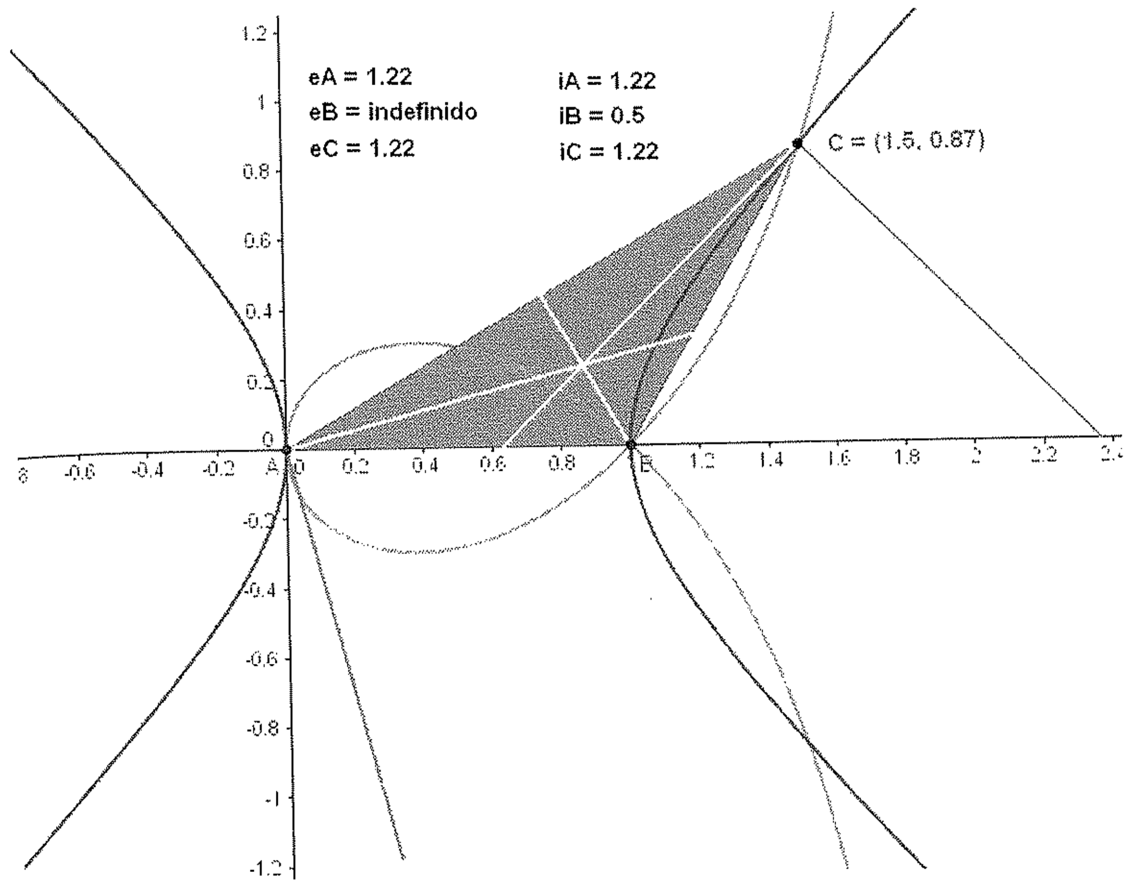

Fig. 5. A triangle with two vertices (C, A) having at each of them internal and external bisectors of equal length. As shown on top of the triangle, it happens that the four lengths are equal to 1.22. Coordinates for C are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

the case of triangles which could happen to have, at two or three vertices, equal lengths for all involved bisectors. For instance, we already know that the three vertices case is impossible.

More appealing seems the following variant. Let us consider now, for each vertex of a triangle, the new (right) triangle defined by the vertex and the feet of the corresponding internal and external bisectors. Let us name such triangles the *bisector triangle* of a given vertex. Since they are always right triangles, two such triangles with coincident bisector lengths will have the same area. So, we might ask, more generally, about the triangles ABC where at least two such bisector triangles have equal area.

Taking $A(0,0)$ and $B(1,0)$ and being AP, AQ, CM, CN the bisectors corresponding to the vertices A and C, we want to find, for example, the locus set of $C(v,w)$ such that $area(A,P,Q) = area(C,M,N)$. With notations as above we take as thesis T:

$$Det(Mat([[v,w,1],[a,b,1],[c,d,1]])) - Det(Mat([[x,y,1],[i,j,1],[g,h,1]]))$$

and then, in $H_c + H_a + T$, all indeterminates but v and w are eliminated. We summarize now the results obtained following this procedure.

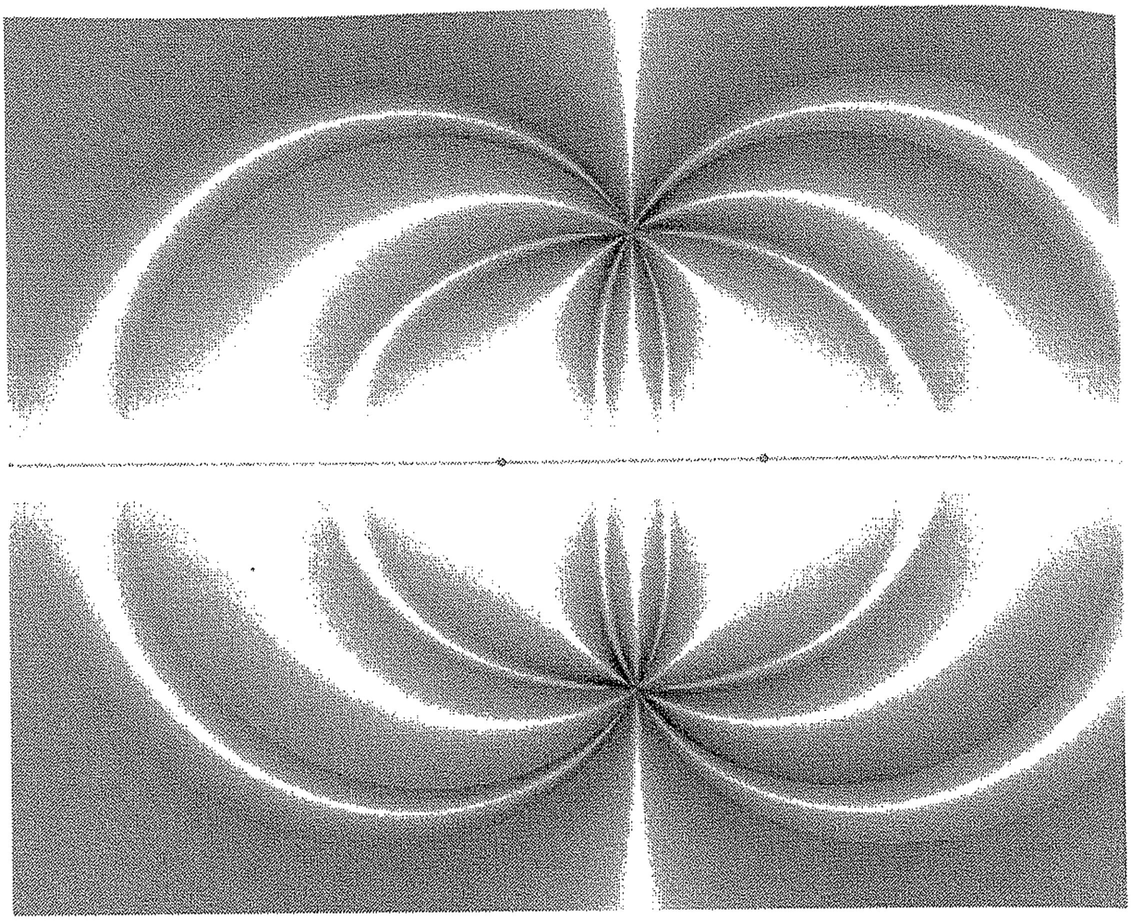

Fig. 6. Graphic result obtained for the problem of equality of the areas tA, tB, tC of triangles determined by the internal and external bisectors at vertices A, B, C. Consider, for each location of C in the plane, the minimum of the absolute values $tA - tB, tB - tC, tC - tA$. Red (gradient) color is associated to this point C when the minimum is reached at $tA - tB$, green if it is at $tB - tC$, blue if at $tC - tA$. Visit http://geogebra.es/imag/fig6 for a full color figure.

– Triangles such that $area(C, M, N) = area(A, P, Q)$

$$G_{AC} = F_{1AC} \times F_{2AC} \times F_{3AC}$$
$$F_{1AC} = w$$
$$F_{2AC} = v^2 + w^2 - 2v$$
$$F_{3AC} = v^4 + 2v^2w^2 + w^4 - v^2 - w^2 + 2v - 1 \text{ (Descartes' oval).}$$

– Triangles such that $area(A, P, Q) = area(B, R, S)$

$$G_{AB} = F_{1AB} \times F_{2AB} \times F_{3AB}$$
$$F_{1AB} = w^2$$
$$F_{2AB} = 2v - 1$$
$$F_{3AB} = v^4 + 2v^2w^2 + w^4 - 2v^3 - 2vw^2 + v^2 + w^2 - 1 \text{ (Cassini's oval)}$$

– Triangles such that $area(C, M, N) = area(B, R, S)$

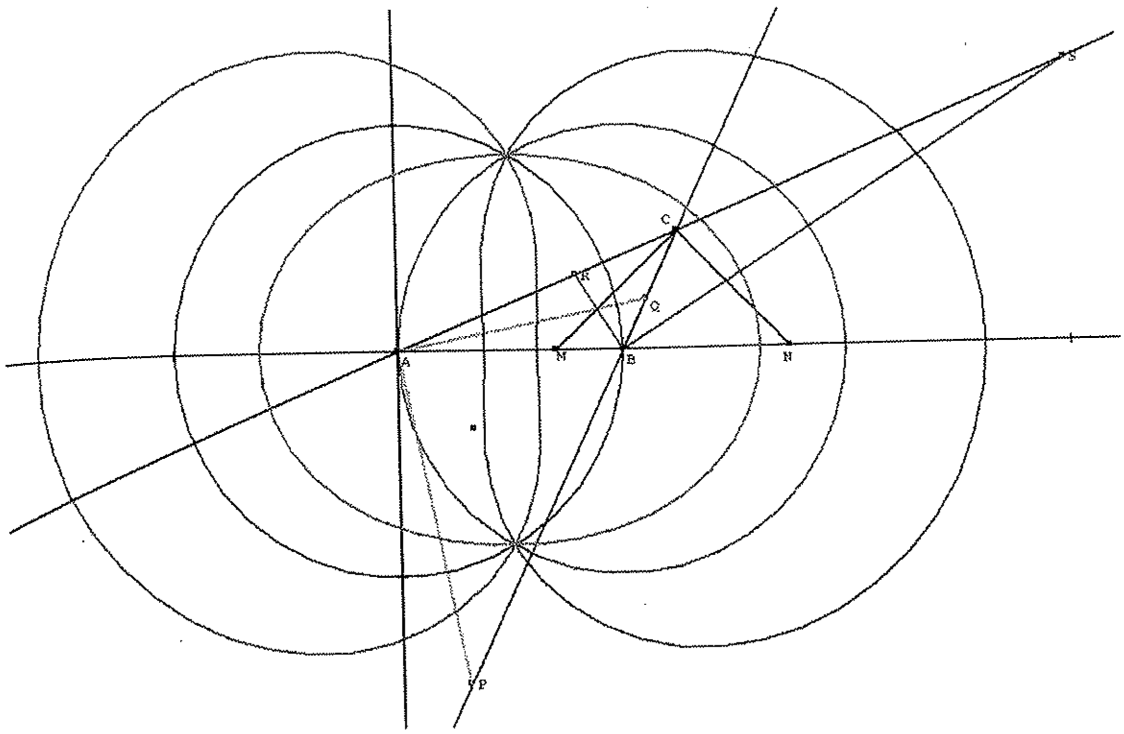

Fig. 7. Results obtained by GDI to the problem of equality of the areas of triangles determined by the internal and external bisectors at each vertex.

$$G_{BC} = F_{1BC} \times F_{2BC} \times F_{3BC}$$
$$F_{1BC} = w^2$$
$$F_{2BC} = v^2 + w^2 - 1$$
$$F_{3BC} = v^4 + 2v^2w^2 + w^4 - 4v^3 - 4vw^2 + 5v^2 + w^2 - 4v + 1 \text{ (Descartes' oval)}$$

Therefore, we conclude that the triangles with base $A(0,0)$ and $B(1,0)$ and verifying the equality of at least one couple of bisector triangles areas, must have the third vertex C on the curve $G_{AC} \times G_{AB} \times G_{BC} = 0$, displayed on [Figure 6] and [Figure 7].

The observation of any of the above figures shows that there are triangles in which $area(C, M, N) = area(B, R, S) = area(A, P, Q)$. It is easy to check that the only two possible solutions are $C=(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, so there are no other than equilateral triangles with equal areas for the three bisector triangles. In fact, it is the case of infinite areas.

6 Conclusion

The usage of Dynamic Geometry programs (DGS), such as GeoGebra, has clear advantages for the study and modeling of different geometric situations. But the purely numerical approach –in practice, the only one available in these programs for the manipulation of large expressions– and the restriction to handle primitive objects (such as lines, points or conics) of these systems, presents serious drawbacks when dealing with automatic discovery of geometric facts. We think that enhancing DGS with symbolic computation features is one way to overcome these limitations.

We have approached a collection of open questions, concerning the equality of lengths for bisectors on a given vertex of a triangle, by using automatic deduction algorithms through a dynamic geometry program (GDI), one of the few including such discovery features. The background symbolic computations have been performed in CoCoA. Moreover, the support of a specific GeoGebra package for displaying complicated locus, has been essential in the exploratory phase of our research and very useful for the graphic presentation of the output.

We believe that our contribution shows the interest, the simplicity and the power of this collaborative approach (GDI, CoCoA, GeoGebra) to the discovery of new geometric results. We have shown how the different tasks can be formulated with a Dynamic Geometry package (GDI), solved with the help of a computer algebra package (CoCoA), and represented through a graphic –numerical– tool developed within a different DGS (GeoGebra). We are working towards the full integration of all these features into a single, widely distributed and performing product that could automatize the different steps and tasks we have performed in our paper.

Acknowledgment

Second author acknowledges the support of the research grant MTM2008-04699-C03- 03 from the Spanish MEC.

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